

The Time Evolution of Quantum Universe in the Quantum Potential Picture

A. Błaut*and J. Kowalski–Glikman[†]

Institute for Theoretical Physics

University of Wrocław

Pl. Maxa Borna 9

Pl-50-204 Wrocław, Poland

February 7, 2008

Abstract

We use the quantum potential approach to analyse the quantum cosmological model of the universe. The quantum potential arises from exact solutions of the full Wheeler-De Witt equation.

PACT number 04.60 Ds

*e-mail address ablaut@ift.uni.wroc.pl

[†]e-mail address jurekk@ift.uni.wroc.pl

1 Introduction

In the recent paper [1] a class of solution of the regularized Wheeler-De Witt equation was given. An interpretation of the resulting "wave functionals of the universe" in terms of the modified field dynamics was also proposed, namely, in the properly extended version of the quantum potential approach to the quantum mechanics given by David Bohm (see e.g. [2],[3],[4]). In this approach the time evolution of the fields is generated by the Hamiltonian of the classical system modified by the additional term, the so called quantum potential. The purpose of the present note is to use the above language to analyse a simple quantum cosmological model.

In the second section we shortly repeat the most important steps leading to the formulation of the quantum theory, essentially quantum gravity, in the quantum potential language. We refer the reader interested in the details to the paper [3]. In what follows we use the notation of [1]. As compared with the work [3] here we have to do with one important modification resulting from the presence of the L_{ab} term in our Wheeler - De Witt operator.

In the third section we will find the equation of motion governing the simplified cosmological model, the homogeneous and isotropic universe. It turns that it has no singular points and the scale factor grows exponentially near the classical singularity $a = 0$.

2 Quantum potential interpretation

We recall that the regularized Wheeler-De Witt operator used in [1] had the form

$$\begin{aligned} \mathcal{H} = & -\kappa^2 \hbar^2 \int dx' K(x, x'; t) G_{abcd}(x') \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} + \\ & + \kappa^2 \hbar^2 L_{ab}(x) \frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2} \sqrt{h(x)} (R(x) + 2\Lambda), \end{aligned} \quad (1)$$

where the function $K(x, x'; t)$ responsible for the point-splitting and the function $L_{ab}(x)$ corresponding to the operator ordering were fixed during the process of regularization and renormalization. The renormalized action of \mathcal{H} on the states was also defined. Our task is to apply the quantum potential approach to the Wheeler-De Witt equation with quantum Hamiltonian (1).

We assume that the wave function of the universe is of the form

$$\Psi = e^\Gamma e^{i\Sigma}, \quad (2)$$

with Γ and Σ the real functionals of the three-metric. Substituting this to the Wheeler–De Witt equation and taking the real part¹ we obtain

$$\begin{aligned} & -\kappa^2 G_{abcd}(x) \frac{\delta \Sigma}{\delta h_{ab}(x)} \frac{\delta \Sigma}{\delta h_{cd}(x)} + \frac{1}{\kappa^2} \sqrt{h(x)} (R(x) + 2\Lambda) + \\ & + L_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} + e^{-\Gamma} \kappa^2 \left(\frac{\delta^2 e^\Gamma}{\delta h^2} \right)_{ren} (x) = 0, \end{aligned} \quad (3)$$

where in the last term we used the abbreviated notation to indicate that the action of the second functional derivative is renormalized. Then we define the momenta as the functional gradient of Σ , to wit

$$p^{ab}(x) = \frac{\delta \Sigma}{\delta h_{ab}(x)}. \quad (4)$$

With this identification (3) turns to the Hamilton-Jacobi equation for general relativity with additional term corresponding to quantum potential:

$$\begin{aligned} & \kappa^2 G_{abcd}(x) p^{ab}(x) p^{cd}(x) - \frac{1}{\kappa^2} \sqrt{h(x)} (R(x) + 2\Lambda) + \\ & - \hbar^2 L_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} - \hbar^2 e^{-\Gamma} \kappa^2 \left(\frac{\delta^2 e^\Gamma}{\delta h^2} \right)_{ren} (x) = 0. \end{aligned} \quad (5)$$

We see that in the limit $\hbar \rightarrow 0$ we obtain the classical hamiltonian constraint. The wave function is subject to the second set of equations, namely the ones enforcing the three dimensional diffeomorphism invariance. These equations read (for imaginary part)

$$\nabla^a \frac{\delta \Sigma}{\delta h_{ab}(x)} = \nabla^a p_{ab} = 0 \quad (6)$$

Thus our theory is defined by two equations (5) and (6). Now we can follow without any alternations the derivation of Gerlach [5] to obtain the full set of

¹In this paper we do not consider the imaginary part. For its interpretation see for example [2].

ten equations governing the quantum gravity theory in the quantum potential approach

$$0 = \mathcal{H}^a = \nabla_a p^{ab}, \quad (7)$$

$$0 = \mathcal{H}_\perp = -\kappa^2 G_{abcd}(x) p^{ab} p^{cd} + \frac{1}{\kappa^2} \sqrt{h(x)} (R(x) + 2\Lambda) \\ + L_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} + \kappa^2 e^{-\Gamma} \left(\frac{\delta^2 e^\Gamma}{\delta h^2} \right)_{ren} (x), \quad (8)$$

$$\dot{h}_{ab}(x, t) = \{h_{ab}(x, t), \mathcal{H}[N, \vec{N}]\}, \quad (9)$$

$$\dot{p}^{ab}(x, t) = \{p^{ab}(x, t), \mathcal{H}[N, \vec{N}]\}. \quad (10)$$

In equations above, $\{\star, \star\}$ is the usual Poisson bracket, and

$$\mathcal{H}[N, \vec{N}] = \int d^3x (N(x) \mathcal{H}_\perp(x) + N^a(x) \mathcal{H}_a(x)) \quad (11)$$

is the total hamiltonian (which is a combination of constraints). The above set of equations describes the time evolution corresponding to a given solution of the Wheeler-De Witt equation of the classical fields $h_{ab}(t)$ and $p^{ab}(t)$ from given initial surface. In this way, in this approach we are able to circumvent the problem of time of quantum gravity [6]. It should be noted that because of the presense of the quantum potential term in the superpotential the algebra of \mathcal{H} doesn't close in general case, so it must be checked for every solution separately.

It might seem puzzling at the first sight why to a single wavefunction there corresponds a set of equations with, clearly, many solutions. The resolution of this problem is that the wavefunction, as a rule, is sensitive only to some aspects of the configuration. For example, the exact wavefunction $\Psi_I = \exp(-\frac{3\rho^{(5)}}{\Lambda} \mathcal{V})$ found in [1] depends on the total volume of the universe, \mathcal{V} , only, and thus any configuration with given volume leads to the same numerical value of it. The above dynamical equations provide us with much more detailed information concerning the dynamics of the system than the wavefunction alone.

3 A simple model

In [1] we have found three real solutions of the Wheeler-De Witt equation. Such states were interpreted as "frozen in time" since real solutions do not,

by definition, evolve in time (cf.[3]). We therefore take the complex superposition of the solutions,

$$\Psi = a\Psi_I + b\Psi_{II}, \quad (12)$$

$$a = |a|e^{i\alpha}, b = |b|e^{i\beta} \quad |a| = |b|,$$

where $\Psi_I = \exp(-\frac{3\rho^{(5)}}{\Lambda}\mathcal{V})$, $\Psi_{II} = \exp(\frac{4}{3}\Lambda\frac{1}{\kappa^4\hbar^2\rho^{(5)}}\mathcal{R})$ are two exact solutions of the Wheeler–De Witt equation found in [1]. Here $\mathcal{V} = \int \sqrt{h}$ is the volume of the universe, $\mathcal{R} = \int \sqrt{h}R^{(3)}$ its average curvature, and $\rho^{(5)}$ is the renormalization constant. Now we can follow the prescription given in the previous section to get the dynamical equation for our system corresponding to the state (12).

The hamiltonian (8) can be computed to be

$$H_{\perp} = \kappa^2 G_{abcd} \pi^{ab} \pi^{cd} + \frac{A\Psi_I^2\Psi_{II}^2}{|\Psi|^4} \left(\frac{27}{16} \frac{\rho^{(5)2}\hbar^2\kappa^2}{\Lambda^2} \sqrt{h} + \frac{1}{\kappa^2} \sqrt{h} R - \frac{8}{9} \frac{\Lambda^2}{\hbar^2\kappa^6\rho^{(5)2}} \sqrt{h} \left(-\frac{3}{8} R^2 + R_{ab}R^{ab} \right) \right), \quad (13)$$

where $A = 2|a|^4 \sin^2(\alpha - \beta)$ is a parameter which measures the rate of mixture of two universes.

We can readily write down the dynamical equations of motion (in the gauge $\vec{N} = 0$ and $N = N(t)$.) Equation (9) takes the form

$$\dot{h}_{ab} = 2N\kappa^2 G_{abcd} \pi^{cd}, \quad (14)$$

which can be solved for π^{ab} in a standard way

$$\pi^{ab} = \frac{\sqrt{h}}{2\kappa^2 N} \left(\dot{h}^{ab} - \text{tr}(\dot{h}) h^{ab} \right), \quad (15)$$

where $\dot{h}^{ab} = h^{ac} h^{bd} \dot{h}_{cd}$, $\text{tr}(\dot{h}) = h^{ab} \dot{h}_{ab}$. Equation (10) takes the form

$$\begin{aligned} \dot{\pi}^{ab} &= \frac{1}{2\kappa^2 N} \left(\ddot{h}_{cd} G^{abcd} + \dot{h}_{cd} \dot{G}^{abcd} - \dot{h}_{cd} G^{abcd} \frac{\dot{N}}{N} \right) = \\ &= \frac{\sqrt{h}}{N\kappa^2} \left[\frac{1}{8} h^{ab} \text{tr}(\dot{h} \times \dot{h}) - \frac{1}{2} (\dot{h} \times \dot{h})^{ab} + \frac{1}{2} \dot{h}^{ab} \text{tr}(\dot{h}) \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{N\sqrt{h}}{\kappa^2}\mathcal{F}\left(\frac{1}{2}h^{ab}R - R^{ab}\right) - \frac{27N\sqrt{h}}{32}\frac{\rho^{(5)2}\hbar^2\kappa^2}{\Lambda^2}\mathcal{F}h^{ab} \\
& + \frac{8N\sqrt{h}}{9}\frac{\Lambda^2}{\hbar^2\kappa^6\rho^{(5)2}}\mathcal{F}\left[\frac{1}{2}h^{ab}R_{cd}R^{cd} - 4R^{c(a}R_c^{b)} - \frac{3}{16}R^2h^{ab} + \frac{3}{4}RR^{ab} + \right. \\
& \quad \left. + \frac{1}{2}\nabla^{ab}R - \square R^{ab} + \frac{1}{4}h^{ab}\square R\right] \\
& - N\int\sqrt{h}\left[\frac{27}{16}\frac{\rho^{(5)2}\hbar^2\kappa^2}{\Lambda^2} + \frac{1}{\kappa^2}R - \frac{8}{9}\frac{\Lambda^2}{\hbar^2\kappa^6\rho^{(5)2}}\left(-\frac{3}{8}R^2 + R_{ab}R^{ab}\right)\right]\frac{\delta\mathcal{F}}{\delta h_{ab}},
\end{aligned} \tag{16}$$

where

$$\mathcal{F} = \frac{1}{2} \frac{\sin^2(\alpha - \beta)}{\left\{ \cosh\left(\frac{3\rho^{(5)}}{\Lambda}\mathcal{V} + \frac{4\Lambda}{3\hbar^2\kappa^4\rho^{(5)}}\mathcal{R}\right) + \cos(\alpha - \beta) \right\}^2} \tag{17}$$

We see that quantum effects exhibit themselves in two ways: first there are higher curvature terms in the effective quantum hamiltonian, and second the resulting coupling constants are modified not only by quantum corrections following from the renormalization, but also by nonlocal terms related to the global structure of the universe. This latter fact was to be expected in the framework based on the quantum potential approach since the quantum potential is usually nonlocal, however it cannot be merely treated as a mere artefact of the method employed. Even if the “reality” of the field evolution described by equations (8), (9), (10) can be questioned in the highly quantum regime, without doubts these equations provide a correct semiclassical approximations, and in this approximation some traces of nonlocality will still be present. It is our opinion however that the nonlocal terms discovered above do reflect the deep structure of quantum gravity. These questions and the analysis of solutions of the dynamical equation in the cosmological context will be subject of the separate paper.

Now we present a solution of the resulting equations assuming that $R_{ab} = 0$, $h_{ab}(t) = a(t)\tilde{h}_{ab}$, where \tilde{h} is a flat metric on compact three-manifold (space) with normalization $\int\sqrt{\tilde{h}} = 1$. The equation of motion for this ansatz that follow from (16) reduce to the following equation:

$$\frac{\ddot{a}}{aN^2} - \frac{1}{4}\frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}}{a}\frac{\dot{N}}{N^3} +$$

$$-B \frac{\sin^2(\alpha - \beta)}{\left[\cosh\left(\frac{3\rho^{(5)}}{\Lambda}a^{3/2}\right) + \cos(\alpha - \beta)\right]^3} \left[\frac{1}{4} - \frac{3\rho^{(5)}}{2\Lambda}a^{3/2} \sinh\left(\frac{3\rho^{(5)}}{\Lambda}a^{3/2}\right)\right] = 0, \quad (18)$$

where $\frac{B}{\kappa^2} = \frac{27}{16} \frac{\rho^{(5)2} \hbar^2 \kappa^2}{\Lambda^2}$. In agreement with what was mentioned above, we see that the limit $\hbar \rightarrow 0$ corresponds to the classical model. The Hamiltonian of the system which is the reduced version of (13) has the form

$$H(x) = \frac{1}{\kappa^2} \left(-\frac{3}{2} \frac{\dot{a}^2}{a^{1/2} N^2} + \frac{B}{2} a^{3/2} \frac{\sin^2(\alpha - \beta)}{\left[\cosh\left(\frac{3\rho^{(5)}}{\Lambda}a^{3/2}\right) + \cos(\alpha - \beta)\right]} \right). \quad (19)$$

It can be easily checked that the evolution (18) is generated by the Hamiltonian (19). It also follows from the form of (19) that after reduction we are left with the reparametrization invariant theory. The solution of (19) near the point $a = 0$ are modified as compared to their classical singular behaviour because of the quantum potential term that acts effectively as a nonzero cosmological constant in that region. The scale factor grows here exponentially

$$a \sim \exp\left(\pm \frac{3}{4} \frac{\kappa^2 \hbar \rho^{(5)}}{\Lambda} t\right). \quad (20)$$

For $a \gg 1$ quantum potential vanishes and we come back to the classical case.

4 Conclusions

In the paper we analysed time evolution of quantum universes that arises as a complex combination of exact solutions of the Wheeler–De Witt equation presented in [1] by making use of the quantum potential interpretation. We noticed that in the simplified situation considered explicitly, the theory is reparametrization invariant contrary to the general case where the quantum potential may spoil the invariance in time direction. We show that in the simplified model considered above the initial cosmic singularity is avoided. It is a matter of future investigations to check as to whether this property is generic for quantum universes described by equations (16) or not.

References

- [1] A. Błaut and J. Kowalski-Glikman *Quantum Gravity in Metric Representation: Constraints, Solutions and their Interpretation*;
- [2] D. Bohm, Phys. Rev. **85**, 166 (1952), Phys. Rev. **85**, 180 (1952); D. Bohm and B.J. Hiley, Phys. Rep. **144**, 323 (1987); D. Bohm, B.J. Hiley, and P.N. Koloyrou Phys. Rep. **144**, 349 (1987); D. Bohm, B.J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory*, Routledge & Kegan Paul, London, 1993; J. S. Bell *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, 1987; P.R. Holland, *The Quantum Theory of Motion*, Cambridge University Press, 1993.
- [3] J. Kowalski-Glikman; *Quantum Potential and Quantum Gravity* in From Field Theory to Quantum Groups, B. Jancewicz and J. Sobczyk eds., World Scientific, 1996; gr-qc 9511014.
- [4] Yu. Shtanov *Pilot Wave Quantum Gravity*, gr-qc 9503005.
- [5] Ulrich H. Gerlach, Phys. Rev. **177**, 1929 (1969).
- [6] C.J. Isham, *Canonical Quantum Gravity and the Problem of Time* in Integral Systems, Quantum Groups, and Quantum Field Theories, L.A. Ibort and M.A. Rodriguez eds., Kluwer Academic Publishers, 1993.